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CONTROL SYNTHESIS FOR A SYSTEM WITH NON-LINEAR RESISTANCE†

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A dynamic controlled system with one degree of freedom is considered. The system is acted upon by a non-linear resistance, a controlling force and a disturbance. The control and the disturbance are of bounded magnitude. A feedback control is designed which takes the system to a prescribed state. The proposed control law has certain advantages compared with the alternative approach which totally ignores the disturbances. The control is synthesized for an arbitrary non-linear resistance. It is time-suboptimal and robust to variations of the parameters and disturbances.

1. STATEMENT OF THE PROBLEM

CONSIDER a system with one degree of freedom whose dynamics is described by the equation

$$mq^{\cdot\cdot} = R(q^{\cdot}) + U + V(q, q^{\cdot}, t) \tag{1.1}$$

Here q is the generalized coordinate of the system, $m>0$ is a constant inertial coefficient (the mass), $R(q^{\cdot})$ is the resistance, U is the control and $V(q, q^{\cdot}, t)$ is the disturbance; a dot denotes differentiation with respect to time t.

We will assume that the resistance $R(q^{\cdot})$ is directed in the opposite direction to the velocity and its magnitude is strictly increasing as the velocity increases; it is zero in the state of rest. Also $R(q^{\cdot})$ is a smooth function. Hence, we have

$$q^{\cdot}R(q^{\cdot}) < 0, \ dR(q^{\cdot})/dq^{\cdot} < 0 \ (q^{\cdot} \neq 0), \quad R(0) = 0 \tag{1.2}$$

The control and the disturbance are assumed to be bounded by geometric constraints, and the maximum disturbance is strictly less than the maximum control. We have

$$|U| \leqslant U_0, \ |V(q, q^{\cdot}, t)| \leqslant \rho U_0, \ \rho < 1 \tag{1.3}$$

Here $U_0>0$ and $\rho<1$ are constants. In all other respects, the disturbance $V(q, q^{\cdot}, t)$ may be an arbitrary function of its arguments.

It is required to construct a feedback control $U(q, q^{\cdot})$ which takes the system (1.1) from an arbitrary initial state

$$q(t_0) = q^0, \ q^{\cdot}(t_0) = (q^{\cdot})^0 \tag{1.4}$$

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to a prescribed final state with zero velocity

$$q(t_1) = q^*, \; q^{\cdot}(t_1) = 0 \tag{1.5}$$

in a finite time. Here t^0, q^0, $(q^{\cdot})^0$, q^* are some given values, the time t_1 is not fixed.

Let $l>0$ be some quantity of the same dimensions as the coordinate q. We introduce the dimensionless variables

$$x = \frac{(q-q^*)}{l}, \quad t' = \frac{(t-t_0)}{T_0}, \quad u = \frac{U}{U_0'}, \quad f = -\frac{R}{U_0'} \tag{1.6}$$
$$v = \frac{V}{U_0}, \quad T_0 = \left(\frac{ml}{U_0}\right)^{1/2}$$

Making the change of variables (1.6) in Eq. (1.1), we obtain

$$x^{\cdot\cdot} + f(x^{\cdot}) = u + v(x, x^{\cdot}, t) \tag{1.7}$$

Here and in what follows, dots denote derivatives with respect to dimensionless time t', which in (1.7) and below is written simply as t. By (1.2) and (1.6) the smooth function $f(x^{\cdot})$ has the following properties:

$$zf(z) > 0, \; f'(z) > 0 \; (z \neq 0), \; f(0) = 0 \tag{1.8}$$

The variables u and v in (1.7) are constrained by [see (1.3) and (1.6)]

$$|u| \leqslant 1, \; |v| \leqslant \rho, \; \rho < 1 \tag{1.9}$$

After the change of variables (1.6), the initial conditions (1.4) and the final conditions (1.5) take the form

$$x(0) = \xi, \; x^{\cdot}(0) = \eta \tag{1.10}$$

$$x(T) = 0, \; x^{\cdot}(T) = 0 \tag{1.11}$$

Here

$$\xi = (q^0 - q^*)/l, \; \eta = (q^{\cdot})^0 T_0/l, \; T = (t_1 - t_0)/T_0$$

Our control problem now can be stated in the following form.

Problem 1. Construct a feedback control $u(x, x^{\cdot})$ which satisfies the constraint (1.9) and takes system (1.7) with an arbitrary disturbance v constrained by (1.9) from any initial state (1.10) to a prescribed final state (1.11) in a finite time.

Note that this formulation is a generalization of the problems discussed in [1, 2] to the case of a non-linear resistance $f(x^{\cdot})$ in the system (1.7). The function $f(x^{\cdot})$ should satisfy conditions (1.8) and in all other respects it is quite arbitrary.

2. A GAME-THEORETICAL APPROACH

Let us consider Eq. (1.7) from the point of view of differential game theory, assuming that u and v are the controls of two opponents constrained by (1.9). We will seek a positional control $u(x, x^{\cdot})$ which takes system (1.7) from state (1.10) to state (1.11) in the shortest guaranteed time T for any admissible disturbance v. The control $u(x, x^{\bullet})$ obtained by solving the differential game produces, as is easily seen, a solution of Problem 1. On the other hand, solution of the differential game reduces [3, 4] to a solution of a time-optimal control problem for the system

$$x^{\bullet\bullet} + f(x^{\bullet}) = (1-\rho)u; \quad |u| \leqslant 1, \ 0 \leqslant \rho < 1, \ T \to \min \tag{2.1}$$

with boundary conditions (1.10) and (1.11). Equation (2.1) is obtained from (1.7) for $v = -\rho u$, which corresponds to the worst (for u) opponent control: the optimal player controls are such that $u = \pm 1$, $v = \pm\rho$ at any instant.

The control $u(x, x^{\bullet})$ required in Problem 1 and the corresponding time T are obtained by synthesizing the time-optimal control for Eq. (2.1) with boundary conditions (1.10) and (1.11). The time-optimal control problem is written in the form

$$x_1^{\bullet} = x_2, \ x_2^{\bullet} = -f(x_2) + (1-\rho)u; \quad |u| \leqslant 1, \ 0 \leqslant \rho < 1 \tag{2.2}$$

$$x_1(0) = \xi, \ x_2(0) = \eta, \ x_1(T) = x_2(T) = 0, \ T \to \min$$

$$(x_1 = x, \ x_2 = x^{\bullet})$$

3. TIME-OPTIMAL CONTROL

We will solve problem (2.2) by the maximum principle. The Hamiltonian for problem (2.2) has the form

$$H = p_1 x_2 + p_2 [(1-\rho)u - f(x_2)] \tag{3.1}$$

where p_1 and p_2 are conjugate variables. The conjugate system has the form

$$p_1^{\bullet} = 0, \ p_2^{\bullet} = -p_1 + f'(x_2)p_2 \tag{3.2}$$

Since (2.2) is an autonomous system, our time-optimal control problem has the first integral

$$H = p_1 x_2 + p_2 [(1-\rho)u - f(x_2)] = h \geqslant 0 \tag{3.3}$$

where h is a constant.

By the maximum principle, we obtain from (3.1) subject to the constraint $|u| \leqslant 1$ [see (2.2)]

$$u = \operatorname{sign} p_2 \tag{3.4}$$

Let us consider the possibility of the existence of singular sections of the optimal trajectory on which $p_2 = 0$. On such a singular section, by the second equation in (3.2), we have also $p_1 = 0$. Therefore, if a singular section exists, we have $p_1 = \text{const} = 0$ on the entire trajectory. But then the second equation in (3.2) is homogeneous on the entire trajectory, and since $p_2 = 0$ on the singular section, we have $p_2 \equiv 0$ on the entire trajectory. However, by the maximum principle, the conjugate vector does not vanish identically on the optimal trajectory. The contradiction proves that the optimal trajectory is free from singular sections. Thus, the equality $p_2 = 0$ may be observed only at isolated instants of time (switching points) and, by (3.4), we have $u = \pm 1$ almost everywhere.

Let us first consider the sections of the optimal trajectory where $p_2 > 0$, $u = 1$. From Eqs (2.2) we obtain for these sections

$$dx_1/dx_2 = x_2 [1 - \rho - f(x_2)]^{-1} \tag{3.5}$$

From (3.5) it follows that in the x_1, x_2 plane the sections of the optimal trajectory with $p_2 > 0$ are arcs of the curves

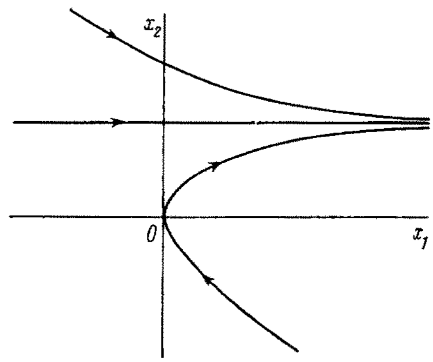

FIG. 1.

$$x_1 = \varphi_\rho^+ (x_2) + c^+ \ (p_2 > 0) \tag{3.6}$$

Here c^+ is an arbitrary constant and the function $\varphi_\rho^+ (x_2)$ is defined by the equality

$$\varphi_\rho^+ (y) = \int_0^y \frac{z\,dz}{1-\rho-f(z)}, \quad 0 \leqslant \rho < 1 \tag{3.7}$$

We note some properties of the function $\varphi_\rho^+ (y)$, that follow from equalities (3.7), (1.8) and are needed in the sequel. As y varies from $-\infty$ to 0, the function φ_ρ^+ is positive and strictly decreasing, vanishing for $y=0$. The point $y=0$ is the unique extremum of the function $\varphi_\rho^+ (y)$ (its minimum). If the transcendental equation for z^+

$$f(z^+) = 1 - \rho \tag{3.8}$$

is unsolvable, i.e. $f(z) < 1-\rho$ for all z, then the function $\varphi_\rho^+ (y)$ is strictly increasing for all $y \geqslant 0$. In this case $\varphi_\rho^+ (y) > 0$ for all $y \neq 0$.

If, however, z^+ is a root of Eq. (3.8), then this root is positive and unique by conditions (1.8). In this case, the function $\varphi_\rho^+ (y)$ is strictly increasing from 0 to ∞ in the interval $y \in (0, z^+)$ and is strictly decreasing for $y > z^+$. A typical curve of the dependence (3.6) in the x_1, x_2 plane for $c^+ = 0$ is shown in Fig. 1 for the case when Eq. (3.8) has a root $z^+ > 0$. The direction in which the time t increases along the trajectory according to the first equation in (2.2) is shown by arrows in Fig. 1.

We similarly consider the sections of the trajectories with $p_2 < 0$. These sections are arcs of the curves

$$x_1 = \varphi_\rho^- (x_2) + c^- \ (p_2 < 0) \tag{3.9}$$

Here, as in (3.6), c^- is an arbitrary constant and the function φ_ρ^- is defined by an equality similar to (3.7):

$$\varphi_\rho^- (y) = \int_0^y \frac{z\,dz}{-(1-\rho)-f(z)}, \quad 0 \leqslant \rho < 1 \tag{3.10}$$

We introduce a transcendental equation for z^- similar to (3.8):

$$f(z^-) = -(1-\rho) \tag{3.11}$$

If Eq. (3.11) does not have a solution z^-, i.e. $f(z)>\rho-1$ for all z, then the function $\varphi_\rho^-(y)$ (3.10) is strictly increasing for $y<0$ and strictly decreasing for $y>0$. Here $\varphi_\rho^-(y)<0$ for all $y\neq 0$.

If z^- is a root of Eq. (3.11), then it is unique and negative ($z^-<0$) by conditions (1.8). In this case the function $\varphi_\rho^-(y)$ is strictly decreasing for $y\in(-\infty, z^-)$, strictly increasing for $y\in(z^-, 0)$, and again strictly decreasing for $y\in(0, \infty)$. As $y\to z^-$, this function tends to $-\infty$, and for $y=0$ it has a local zero maximum. A typical graph of the function $\varphi_\rho^-(y)$ can be obtained from the graph of the function $\varphi_\rho^+(y)$ in Fig. 1 by a central symmetry transformation (or, equivalently, by simultaneously reversing the directions of both axes a_1, x_2).

The curves described above are the trajectories corresponding to $p_2>0$ and $p_2<0$ that pass through the origin in the x_1, x_2 plane. Other curves whose arcs may be sections of optimal trajectories are obtained from these curves by parallel translation by c^+, c^- along the x_1 axis [see (3.6) and (3.9)].

Note that if the transcendental equations (3.8) and (3.11) have solutions, then system (2.2) has the corresponding solutions

$$x_2 = z^+ \ (p_2>0),\ x_2 = z^- \ (p_2<0) \tag{3.12}$$

In the x_1, x_2 plane, the solutions (3.12) correspond to phase trajectories in the form of straight lines parallel to the x_1 axis. These lines are the asymptotes of the curves (3.6) and (3.9), respectively (see Fig. 1).

Thus, the required optimal trajectories consist of sections of the curves (3.6) and (3.9) with various c^+, c^- and also possibly segments of the straight lines (3.12) if the corresponding equations (3.8) and (3.11) are solvable.

We will now show that each optimal trajectory has at most one control switching point, i.e. the function $p_2(t)$ vanishes at most once.

Assume that this is not so. Suppose that the function $p_2(t)$ vanishes at two instants t', t'', and between them it is positive. Then

$$p_2(t)>0,\ t\in(t',\ t'');\ p_2(t') = p_2(t'') = 0 \tag{3.13}$$

From the first integral (3.3) for t', t'' we obtain by (3.13)

$$p_1x_2(t') = p_1x_2(t'') = h \geqslant 0 \tag{3.14}$$

If $p_1 = \text{const} = 0$, then from (3.2) we obtain for $p_2(t)$ a linear homogeneous equation, which with zero conditions (3.13) at t', t'' has an identically zero solution $p_2(t)\equiv 0$. But this contradicts the maximum principle, which asserts the existence of a nonzero conjugate vector. Therefore, $p_1 = \text{const}\neq 0$ and from (3.14) we obtain $x_2(t') = x_2(t'')$. However, on all phase trajectories except the straight lines (3.12) the variable x_2 is either strictly increasing or strictly decreasing as the time t increases. This follows from the previous analysis of the phase trajectories and is clear in Fig. 1. The equality $x_2(t') = x_2(t'')$ is therefore possible only if the relevant section of the trajectory is a segment of the straight line (3.12), i.e.

$$x_2(t) \equiv z^+,\ \ t\in(t',\ t'') \tag{3.15}$$

Substituting (3.15) in the second conjugate equation (3.2), we obtain a linear equation with constant coefficients

$$p_2^{\cdot} = -p_1 + kp_2,\ \ k = f'(z^+)>0$$

where $k>0$ by (1.8). The general solution of this equation has the form

$$p_2(t) = p_1/k + Ce^{kt} \tag{3.16}$$

where $p_1 = \text{const}\neq 0$ and C is an arbitrary constant. But the solution (3.16) is monotone in t and cannot satisfy conditions (3.13) for any $p_1\neq 0$ and C. Thus, the section of the optimal trajectory where conditions (3.13) hold

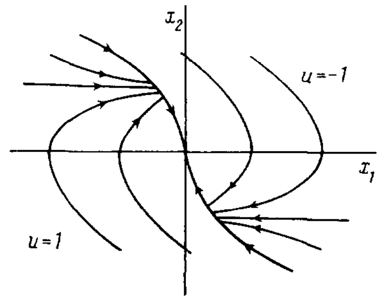

FIG. 2.

cannot be a straight segment of the line (3.15). We have thus shown that an optimal trajectory may not include sections of the form (3.13).

We can similarly prove that an optimal trajectory may not include sections such that the function $p_2(t)$ is negative inside the section and vanishes at its endpoint.

Therefore, on each optimal trajectory the function $p_2(t)$ vanishes at most once, i.e. the control may have at most one switching point.

The only phase trajectories that reach the origin as the time increases are the branch of the curve (3.6) with $c^+ = 0$ which lies in the quadrant $x_1 \geqslant 0$, $x_2 \leqslant 0$ (Fig. 1) and the branch of the curve (3.9) with $c^- = 0$ which lies in the quadrant $x_1 \leqslant 0$, $x_2 \geqslant 0$. These curve branches correspond to the controls $u = 1$ and $u = -1$, respectively. The collection of these branches form the switching curve, whose equation is written as

$$x_1 = \psi_\rho (x_2) \tag{3.17}$$

Here we have introduced the notation

$$\psi_\rho (y) = \varphi_\rho^+ (y), \; y \leqslant 0; \; \psi_\rho (y) = \varphi_\rho^- (y), \; y \geqslant 0 \tag{3.18}$$

By the properties of the functions (3.7) and (3.10), the function $\psi_\rho(y)$ (3.18) is strictly decreasing for all y and vanishes for $y = 0$, where it has a point of inflection.

We can now easily describe the entire field of optimal trajectories. An optimal trajectory originating from any point of the phase plane x_1, x_2 consists of a straight-line segment of one of the families (3.6) or (3.9) and a section of the switching curve (3.17).

The field of optimal trajectories is qualitatively shown in Fig. 2 for the case when Eqs (3.8) and (3.11) have roots. The thick curve in Fig. 2 is the switching curve (3.17) and the arrows indicate the direction of increase of the time t. Note that this picture of the field of optimal trajectories is similar to the picture observed with a linear resistance [6].

The optimal control corresponding to this field of phase trajectories may be represented in the form

$$\begin{gathered} u_\rho (x_1, x_2) = \operatorname{sign} [\psi_\rho (x_2) - x_1], \; x_1 \neq \psi_\rho (x_2) \\ u_\rho (x_1, x_2) = \operatorname{sign} x_1 = -\operatorname{sign} x_2, \; x_1 = \psi_\rho (x_2) \\ (x_1 = x, \; x_2 = \dot{x}) \end{gathered} \tag{3.19}$$

where the function ψ_ρ is defined by relationships (3.18), (3.7) and (3.10).

The control law (3.19) by construction solves Problem 1. This solution may be called suboptimal,

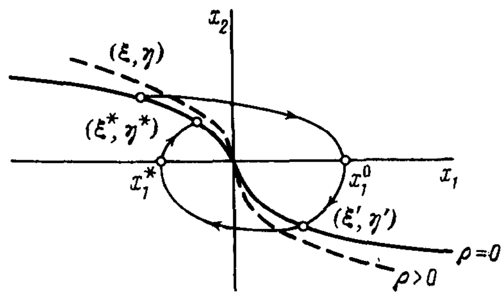

FIG. 3.

because it is time-optimal (unimprovable) when v is the "worst-case" disturbance, as assumed in the game-theoretical approach. With worst-case disturbance $v = -\rho v$, the system moves along optimal trajectories (see Fig. 2). If the disturbance deviates from the worst case ($v \neq -\rho v$), which is usually so, the trajectories deviate from the optimal trajectories. The motion along the switching curve is in the sliding mode, and the time taken to reach the origin only diminishes.

4. SIMPLIFIED APPROACH

So far, we have assumed that the disturbance is unknown but its maximum attainable value is known and essentially affects the synthesized control. In dimensionless variables, the disturbance bound has the form $|v| \leqslant \rho$ [see (1.9)] and the synthesized control (3.19) depends on the parameter ρ.

We often use a different approach to control synthesis in the presence of disturbances, which simply ignores the disturbances. In our case, this simplified approach means that the parameter ρ is set equal to zero during control synthesis and the disturbances are ignored. The control $u_0(x_1, x_2)$ obtained in this way is defined by relationships (3.19), (3.18), (3.7) and (3.10) with $\rho = 0$. The switching curve for simplified control is given by (3.17) with $\rho = 0$. It is represented in Fig. 3 by the thick solid curve. The dashed curve shows, for comparison, the switching curve with $\rho \geqslant 0$.

Let us compare the two control synthesis techniques—the game-theoretical and the simplified method. To this end, we will examine the dynamics of system (1.1) for some $\rho \in (0, 1)$ under the action of the simplified control $u_0(x_1, x_2)$. We will represent this system in the form

$$x_1^{\cdot} = x_2, \; x_2^{\cdot} = -f(x_2) + u_0(x_1, x_2) + v \qquad (4.1)$$
$$|v| \leqslant \rho < 1 \quad (x_1 = x, \; x_2 = x^{\cdot})$$

For system (4.1), we consider the following auxiliary problem of finding the worst-case disturbance.

Problem 2. Find the optimal control $v(x_1, x_2)$ of system (4.1) which satisfies the constraint $|v| \leqslant \rho$ and such that the first intersection of any phase trajectory of this system with the switching curve $x_1 = \varphi_0(x_2)$ lies as far as possible from the origin, i.e. at the maximum possible $|x_1|$ or, equivalently, the maximum possible $|x_2|$.

First assume that the starting point is in the region $x_1 > \psi_0(x_2)$. Then, by (3.19), we have $u_0 = -1$ for the given trajectory. The phase trajectory of system (4.1) first crosses that branch of the curve $\tau_1 = \psi_0(x_2)$ where $x_1 > 0$, $x_2 < 0$ (see Fig. 3). Problem 2 is described by the relationships

$$\begin{gathered} x_1^{\cdot} = x_2, \quad x_2^{\cdot} = -f(x_2) - 1 + v, \quad |v| \leqslant \rho < 1 \\ x_1(0) = \xi, \quad x_2(0) = \eta, \quad \xi > \psi_0(\eta) \\ x_1(\tau) = \psi_0(x_2(\tau)), \quad x_1(\tau) > 0, \quad x_2(\tau) < 0, \quad x_1(\tau) \to \max \end{gathered} \tag{4.2}$$

The instant τ when the process terminates is not fixed. Maximization of $x_1(\tau)$ is equivalent by (4.2) to minimization of the integral functional

$$J = \int_0^\tau (-x_2)\,dt \to \min \tag{4.3}$$

Applying the maximum principle to Problems (4.2) and (4.3), we form the Hamiltonian

$$H = p_1 x_2 + p_2 [v - 1 - f(x_2)] + x_2 \tag{4.4}$$

where p_1, p_2 are the conjugate variables. They satisfy the conjugate system

$$p_1^{\cdot} = 0, \quad p_2^{\cdot} = f'(x_2)\,p_2 - p_1 - 1 \tag{4.5}$$

and the transversality conditions corresponding to the boundary conditions (4.2):

$$p_1 \psi_0'(x_2) + p_2 = 0, \quad H = 0 \quad (t = \tau) \tag{4.6}$$

From the first condition in (4.6), applying relationships (3.18) and (3.7) for $\rho = 0$ and noting that $x_2(\tau) < 0$ by (4.2), we obtain

$$p_1 = -p_2 [1 - f(x_2)]/x_2 \quad (t = \tau) \tag{4.7}$$

Substituting (4.7) into (4.4) and using the second transversality condition in (4.6), we obtain after simplifications

$$H = p_2 (v - 2) + x_2 = 0 \quad (t = \tau)$$

Since $x_2(\tau) < 0$ and $|v| \leqslant \rho < 1$, we obtain from this equality

$$p_2(\tau) < 0 \tag{4.8}$$

We find the optimal control from the condition for maximum H (4.4) over $|v| \leqslant \rho$,

$$v = \rho \operatorname{sign} p_2 \tag{4.9}$$

Singular sections of the trajectory are ruled out. Indeed, if $p_2 \equiv 0$ in some time interval, then in this interval $p_1 = -1$ by the second equation (4.5). But $p_1 \equiv \text{const}$, and therefore $p_1 = -1$ on the entire trajectory. Then the second equation in (4.5) becomes linear and homogeneous for p_2 and its solution with initial condition (4.8) does not vanish. Thus, there are no singular sections and equality (4.9) implies that the control $v(t)$ has switching points when $p_2(t) = 0$.

Let us find the switching curve in the x_1, x_2 plane. Since system (4.2) is autonomous, its Hamiltonian (4.4) preserves a constant value along the optimal trajectory, and by (4.6) this constant value is zero:

$$H = (p_1 + 1)\,x_2 + p_2 [v - 1 - f(x_2)] \equiv 0$$

Hence it follows that at the switching point, i.e. for $p_2 = 0$, we have either $p_1 = -1$ or $x_2 = 0$. But the inequality $p_1 = -1$, as we have shown, implies that p_2 never vanishes. We thus have $x_2 = 0$ at the switching point, and the switching curve in this case is the ray $x_2 = 0$, $x_1 > 0$.

In order to determine the sign of the control for $x_2 < 0$ and $x_2 > 0$, it suffices to determine its sign at a single point. At the terminal time τ we have $x_2(\tau) < 0$ by (4.2) and $p_2(\tau) < 0$ by (4.8). Thus, $v = -\rho$ for $x_2 < 0$.

As a result,

$$v(x_1, x_2) = \rho \operatorname{sign} x_2 \tag{4.10}$$

We have synthesized the optimal control in the region $x_1 > \psi_0(x_2)$. To obtain the control in the region $x_1 < \psi_0(x_2)$, we note some symmetry properties. When $f(z)$ is replaced with $g(z) = -f(-z)$, we have by (3.7) and (3.10)

$$\varphi_\rho^+(y) \to -\varphi_\rho^-(-y),\ \varphi_\rho^-(y) \to -\varphi_\rho^+(-y),\ (f(z) \to -f(-z)) \tag{4.11}$$

From (3.18) and (4.11) it follows that after this change

$$\psi_\rho(y) \to -\psi_\rho(-y)\ \ (f(z) \to -f(-z)) \tag{4.12}$$

Let us now make in (4.1) the change of variables

$$x_1 \to -x_2,\ x_2 \to -x_1,\ v \to -v,\ f(z) \to -f(-z) \tag{4.13}$$

By (4.12) and (3.19), u_0 is changed to $-u_0$ and system (4.1) remains invariant. Hence it follows that in the region $x_1 < \varphi_0(x_2)$ the field of optimal trajectories and the optimal control are the same as in the region $x_1 > \varphi_0(x_2)$, but with $f(z)$ replaced by $g(z) = -f(-z)$. Since (4.10) is independent of the specific form of the function $f(z)$, it also applies in the region $x_1 < \varphi_0(x_2)$. Thus, equality (4.10) defines the solution of Problem 2 in the entire x_1, x_2 plane.

5. ANALYSIS OF THE PHASE TRAJECTORIES

Consider the motion of system (4.1) under the action of the simplified control $u_0(x_1, x_2)$ defined by relationships (3.19), (3.18), (3.7) and (3.10) for $\rho = 0$ and the worst-case disturbance v from (4.10). Assume that the initial point ξ, η lies on the branch of the switching curve $x_1 = \varphi_0(x_2)$, where $x_1 < 0$, $x_2 > 0$ (see Fig. 3). Let us investigate the phase trajectory until its next intersection with the same branch of the switching curve. This piece of the trajectory consists of four sections, each with constant u_0 and v. These sections have the following endpoints and controls (see Fig. 3):

$$\begin{aligned}
&1)\ (\xi, \eta) \to (x_1^0, 0),\ u_0 = -1,\ v = \rho \\
&2)\ (x_1^0, 0) \to (\xi', \eta'),\ u_0 = -1,\ v = -\rho \\
&3)\ (\xi', \eta') \to (x_1^*, 0),\ u_0 = 1,\ v = -\rho \\
&4)\ (x_1^*, 0) \to (\xi^*, \eta^*),\ u_0 = 1,\ v = \rho
\end{aligned} \tag{5.1}$$

The parameters of the endpoints (5.1) satisfy relationships that reflect their position on the switching curve and on the coordinate axes (see Fig. 3):

$$\begin{aligned}
&\xi = \psi_0(\eta),\ \eta > 0,\ \xi < 0,\ x_1^0 > 0 \\
&\xi' = \psi_0(\eta'),\ \eta' < 0,\ x_1^* < 0 \\
&\xi^* = \psi_0(\eta^*),\ \eta^* > 0,\ \xi^* < 0
\end{aligned} \tag{5.2}$$

Substituting u_0 and v from (5.1) into Eq. (4.1) and integrating along the corresponding sections of the trajectory, we have

$$\xi'-\xi=\int_{\eta}^{0}\frac{z\,dz}{-1+\rho-f(z)}+\int_{0}^{\eta'}\frac{z\,dz}{-1-\rho-f(z)}$$

$$\xi^*-\xi'=\int_{\eta'}^{0}\frac{z\,dz}{1-\rho-f(z)}+\int_{0}^{\eta^*}\frac{z\,dz}{1+\rho-f(z)}$$

Replacing ξ, ξ' and ξ'' by their expressions from (5.2) and using formulas (3.18), (3.7) and (3.10) for $\rho=0$, we obtain

$$\int_{0}^{\eta'}\frac{z\,dz}{1-f(z)}-\int_{0}^{\eta}\frac{z\,dz}{-1-f(z)}=\int_{0}^{\eta}\frac{z\,dz}{1-\rho+f(z)}-\int_{0}^{\eta'}\frac{z\,dz}{1+\rho+f(z)}$$
$$\int_{0}^{\eta^*}\frac{z\,dz}{-1-f(z)}-\int_{0}^{\eta'}\frac{z\,dz}{1-f(z)}=\int_{0}^{\eta'}\frac{z\,dz}{-1+\rho+f(z)}+\int_{0}^{\eta^*}\frac{z\,dz}{1+\rho-f(z)} \tag{5.3}$$

Recall that $\eta'<0$, $\eta>0$, $\eta^*>0$ by (5.2). Set $\eta'=-\eta^0$, $\eta^0>0$ and transform relationships (5.3) so that they contain only integrals over intervals on the positive half-line. Simplifying, we obtain

$$\Phi_4(\eta^0)=\varkappa^2(\rho)\Phi_1(\eta),\quad \Phi_2(\eta^*)=\varkappa^2(\rho)\,\Phi_3(\eta^0) \tag{5.4}$$

Here

$$\Phi_1(y)=\Phi^+(y;f),\quad \Phi_2(y)=\Phi^-(y;f)$$
$$\Phi_3(y)=\Phi^+(y;g),\quad \Phi_4(y)=\Phi^-(y;,g)$$
$$\Phi^{\pm}(y;h)=\int_0^y\frac{z\,dz}{(1+h)[1\pm(1\mp\rho)^{-1}h]} \tag{5.5}$$
$$f=f(z)\geqslant 0,\quad g=-f(-z)\geqslant 0$$
$$\varkappa(\rho)=[\rho(1+\rho)]^{1/2}[(1-\rho)(2+\rho)]^{-1/2}$$

Consider the transcendental equations (5.4) which determine η^0 and η^* for given $\eta>0$ and $\rho\in(0,1)$. To this end, we will note some properties of the functions Φ_i, $i=1,2,3,4$, from (5.5). Recall that by (1.8) $f(z)>0$ for $z>0$ and $f(z)\to 0$ as $z\to 0$.

The denominators in the integrands for the functions Φ_1 and Φ_3 in (5.5) are positive for all $z\geqslant 0$. Therefore, *the functions Φ_1 and Φ_3 are defined and bounded for all $y\geqslant 0$.*

If the equations

$$f(z_2)=1+\rho,\quad g(z_4)=-f(-z_4)=1+\rho \tag{5.6}$$

have solutions for z_2, z_4, then the denominators of the integrands of the corresponding functions Φ_2, Φ_4 in (5.5) vanish for finite z_2, z_4 equal to the roots of the equations (5.6). In this case, Φ_2, Φ_4 are monotone increasing and go to infinity at $y=z_2$ and $y=z_4$, respectively. If Eqs (5.6) have no solutions, then the functions Φ_2, Φ_4 are defined for all $y\geqslant 0$. In both cases, the denominators of the integrands for the functions Φ_2, Φ_4 have maxima over $f\geqslant 0$ and $g\geqslant 0$, which are respectively equal to $(2+\rho)^2(1+\rho)^{-1/4}$. We thus have the inequalities

$$\Phi_2(y)\geqslant \nu y^2/2,\quad \Phi_4\geqslant \nu y^2/2,\quad \nu=4(1+\rho)(2+\rho)^{-2}$$

The functions Φ_2 and Φ_4 are thus always positive and strictly increasing, taking all values from 0 to ∞ for $y\geqslant 0$.

Hence it follows that the transcendental equations (5.4) for any $\eta>0$ and $\rho\in(0, 1)$ have unique positive solutions $\eta^0>0$ and $\eta^*>0$. These solutions are continuous and monotone functions of η.

Let us differentiate equalities (5.4) with respect to η. After simple reductions we obtain

$$\frac{d\eta^*}{d\eta}=\frac{\varkappa^2(\rho)\,\Phi_3'(\eta^0)}{\Phi_2'(\eta^*)}\frac{d\eta^0}{d\eta}=\frac{\varkappa^4(\rho)\,\Phi_3'(\eta^0)\,\Phi_1'(\eta)}{\Phi_2'(\eta^*)\,\Phi_4'(\eta^0)} \tag{5.7}$$

From relationships (5.5) and properties (1.8) of the function $f(z)$, we obtain the inequalities

$$\frac{\Phi_1'(y)}{\Phi_2'(y)}<1,\quad \frac{\Phi_3'(y)}{\Phi_4'(y)}<1,\quad y>0$$

Using the second inequality, we obtain from (5.7)

$$\frac{d\eta^*}{d\eta}=\varkappa^4(\rho)\frac{\Phi_1'(\eta)}{\Phi_2'(\eta^*)},\quad \eta>0 \tag{5.8}$$

We can verify that the function $\varkappa^2(\rho)$ (5.5) is strictly increasing from 0 to ∞ on $\rho\in[0, 1]$, and $\varkappa=1$ for ρ equal to

$$\rho^*=(5^{1/2}-1)/2=0{,}618\ldots \tag{5.9}$$

The number ρ^* is the well-known golden section.

First assume that $\rho<\rho^*$ and therefore $\varkappa^2(\rho)<\alpha$, where $\alpha<1$ is a positive number. Then from (5.8) we have

$$d\eta^*/d\eta<\alpha^2\Phi_1'(\eta)/\Phi_2'(\eta^*),\ \eta>0 \tag{5.10}$$

and hence

$$\Phi_2(\eta^*)<\alpha^2\Phi_1(\eta),\ \eta>0 \tag{5.11}$$

We will show that $\eta^*<\eta$. Assume that this is not so, specifically $\eta^*\geqslant\eta$. From (5.5) we obtain $\Phi_2(y)>\Phi_1(y)$ for all $y=0$. Then, by the monotonicity of the function $\Phi_2(y)$, we obtain the chain of inequalities

$$\Phi_2(\eta^*)\geqslant\Phi_2(\eta)>\Phi_1(\eta)$$

which leads to a contradiction with inequality (5.11). Thus, $\eta^*<\eta$.

Let us transform inequality (5.10), substituting the expressions for the derivatives Φ_1', Φ_2' from (5.5) and using the positivity of the function $f(z)$:

$$\frac{d\eta^*}{d\eta}<\frac{\alpha^2\eta\,[1+f(\eta^*)]\,[1-(1+\rho)^{-1}f(\eta^*)]}{\eta^*\,[1+f(\eta)]\,[1+(1-\rho)^{-1}f(\eta)]}<\frac{\alpha^2\eta\,[1+f(\eta^*)]}{\eta^*\,[1+f(\eta)]}\quad(\eta>0)$$

We can simplify the last inequality, noting that $f(\eta^*)<f(\eta)$ by the monotonicity of $f(z)$ and by the inequality $\eta^*<\eta$. We obtain

$$d\eta^*/d\eta<\alpha^2\eta/\eta^*,\ \eta>0$$

Integrating this inequality with $\eta^*=0$ when $\eta=0$, we obtain $(\eta^*)^2<\alpha^2\eta^2$ or $\eta^*/\eta<\alpha$.

Thus, if $\rho<\rho^*$, where ρ^* is defined in (5.9), then $\eta^*/\eta<\alpha$, i.e. the phase trajectory approaches the origin. The distance from the origin diminishes at a rate not slower than a geometrical progression. The system therefore reaches the prescribed state in a finite time, although after infinitely many control switchings.

Suppose that the system has reached a small neighbourhood of the origin, so that η is sufficiently small. Here η^0 and η^* are also small in view of their continuous dependence on η. Since $f(z)\to0$ as $z\to0$ by (1.8), the terms $f(z)$, $g(z)$ can be omitted in the integrals (5.5) for small y, which gives in the limit

$$\Phi_i(y)\sim y^2/2,\ y\to0,\ i=1, 2, 3, 4$$

The transcendental equation (5.4) for small η thus take the form

$$(\eta^0)^2 = \varkappa^2(\rho)\eta^2, \quad (\eta^*)^2 = \varkappa^2(\rho)(\eta^0)^2$$

Hence we obtain

$$\eta^*/\eta = \varkappa^2(\rho) \tag{5.12}$$

Let $\rho > \rho^*$ and therefore $\varkappa^2(\rho) > 1$. Then, by (5.13), we obtain $\eta^* > \eta$ and the phase trajectory, even if it has reached a small neighbourhood of the origin, eventually moves away from the origin. The system does not go to the prescribed state.

Thus, with an arbitrary function $f(z)$ that satisfies condition (1.8), the simplified approach produces a control $u_0(x_1, x_2)$ which is defined by relationships (3.19) for $\rho = 0$ and has the following properties.

If $\rho < \rho^* = 0.618$, then for any admissible disturbance $|v| \leq \rho$ the system reaches the origin. The time to reach the origin is finite, although the number of switchings in general is infinite.

If $\rho > \rho^*$, there exists an admissible disturbance v defined by equality (4.10) for which the system never reaches the origin.

Therefore, simplified control guarantees a solution of Problem 1 only for $\rho < \rho^*$, i.e. when the ratio of the maximum allowed disturbance to the maximum allowed control does not exceed the golden section.

Specifying the form of the function $f(z)$, we can construct a more detailed picture of phase motion. Note that the case of zero resistance $f(z) = 0$ has been previously considered in detail in [1] and the case of linear resistance $f(z) = \lambda$, $\lambda > 0$, has been considered in [2].

6. CONCLUSION

The proposed control law (3.19) based on the game-theoretical approach takes the given system (1.7) to the origin in a finite time for any non-linearity $f(z)$ and any uncertain disturbance if $\rho < 1$. This control law does not require a knowledge of the disturbance; we only need to know the maximum allowed disturbance, which must not exceed the maximum control.

Let us stress the difference in the requirements imposed on the functions $f(z)$ and $v(x, x^\bullet, t)$. Both these functions may be arbitrary in the framework of the corresponding conditions: (1.8) for $f(z)$ and (1.9) for v. However, the non-linear resistance function $f(z)$ should be known in order to synthesize the control, while the disturbance $v(x, x^\bullet, t)$ is not needed.

The simplified approach to control synthesis, which totally ignores the disturbances, is less effective. It *a priori* takes the system to the origin only for $\rho < \rho^* = 0.618$. If $\rho > \rho^*$, then there exists a disturbance for which the system never reaches the origin.

Yet both approaches have a similar structure and differ only by their switching curves.

The proposed control technique is robust to various disturbances and parameter variations. These factors can be easily incorporated in the analysis if we increase the assumed level of allowed disturbances, i.e. the parameter ρ, creating a certain safety margin by this parameter.

Note that the synthesized control is suboptimal in the sense that it is time-optimal with the worst-case disturbance.

Our results can be applied to various dynamic systems, e.g. to control the electric motors of robotic systems (see [2]). This opens up the possibility of taking into account various resistance laws that are often encountered in practice.

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STATIONARY AND STATIONARIZABLE REGIMES IN NORMAL STOCHASTIC DIFFERENTIAL SYSTEMS†

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Narrow-sense stationary regimes are considered for multi-dimensional non-linear systems described by Ito stochastic differential equations with Wiener processes. The conditions for the existence of stationary and stationarizable one-dimensional distributions are derived. Exact expressions are obtained for stationary distributions in some mechanical systems.

1. MANY problems of statistical dynamics of servo systems and systems with ideal stochastic holonomic and non-holonomic constraints acted upon by position conservative and non-conservative, accelerating and dissipative, gyroscopic forces and disturbances can be reduced to normal stochastic systems by augmenting the state vector [1–3]. A normal stochastic differential system (SDS) is a stochastic system whose state is described by an Ito stochastic differential equation with an appropriate initial condition

$$\mathbf{Z}^{\cdot} = \mathbf{a}(\mathbf{Z}, t) + \mathbf{b}(\mathbf{Z}, t)\,\mathbf{V}, \quad \mathbf{Z}(t_0) = \mathbf{Z}_0 \tag{1.1}$$

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